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STATE OF STRESS OF A STRIP (BEAM) WITH A RECTILINEAR THIN-WALLED INCLUSION

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The influence of a rectilinear thin-walled isotropic inclusion of finite length on the state of stress of a strip (beam) is studied. A system of two singular integro-differential Prandtl-type equations is obtained, whose solution is suitable for an inclusion of any stiffness: from absolutely rigid or flexible but inextensible, to absolutely pliable (slit). Thus, a relation is constructed between the theory of cracks and the theory of thin-walled elastic inclusions. Formulas are presented for the stress distribution in the neighborhood of the end of the thin-walled inclusion.

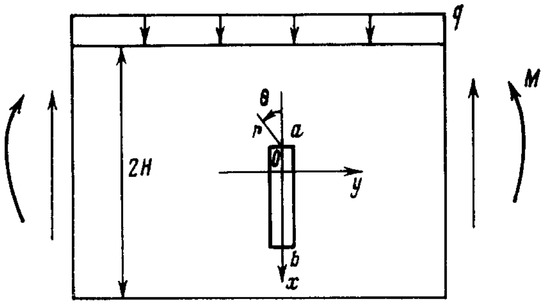

Fig. 1

1. Let us consider an isotropic elastic strip (beam) weakened by a thin-walled elastic inclusion directed perpendicularly to the side faces of the strip (Fig. 1). Let $2H$ and 2τ , respectively, be the width and thickness of the strip, and $2l$ and $2h$ the length and width of the inclusion. We introduce a rectangular Cartesian coordinate system and assume that the inclusion is along the ox-axis at $a \leqslant x \leqslant b$ and $-h \leqslant y \leqslant h$ in the xoy plane, Let external loads in the middle plane of the strip act on this strip, and let the faces of the strip parallel to the xoy plane be assumed free of external stresses.

The quantities characterizing the thin-walled inclusion will be denoted with a zero subscript. The plus and minus superscripts will denote the boundary values of the functions for $y \to +0$ and $y \to -0$, respectively. We denote the segment $[a, b]$ of the real axis by L.

The following boundary conditions hold on the edges of the inclusion

$$(\sigma_y - i\tau_{xy})_0^\pm = (\sigma_y - i\tau_{xy})^\pm, \quad (u + iv)_0^\pm = (u + iv)^\pm \tag{1.1}$$

Following [1], let us consider the strip as an unbounded plate, then the components σ_x, σ_y, τ_{xy} of the stress tensor and the components u and v of the displacement vector are expressed, under the condition of the plane problem of elasticity theory, in terms of two analytic functions $\Phi(z)$ and $\Omega(z)$ by the following formulas:

$$\begin{aligned}
&\sigma_x + \sigma_y = 2\,[\Phi(z) + \overline{\Phi(z)}] \\
&\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar z) + (z - \bar z)\,\overline{\Phi'(z)} \\
&2\mu\,(u' + iv') = \varkappa\Phi(z) - \Omega(\bar z) - (z - \bar z)\,\overline{\Phi'(z)}
\end{aligned} \tag{1.2}$$

For the problem formulated, let us first examine two auxiliary functions of the form

$$\Phi_0(z) = A_0 z^n + A_1 z^{n-1} + \ldots + A_n,\ \Omega_0(z) = B_0 z^n + B_1 z^{n-1} + \ldots + B_n \tag{1.3}$$

which determined the state of stress in a strip without an inclusion depending on the value of the coefficients A_j and B_j $(j = 0, 1, \ldots, n)$.

Neglecting quantities of higher order of smallness as compared to hR, we have for a thin-walled inclusion on the basis of (1.2)

$$\begin{aligned}
&(\sigma_y - i\tau_{xy})_0^+ + (\sigma_y - i\tau_{xy})_0^- = \\
&\qquad \frac{2}{(1+\varkappa_0)}\,[(1-\varkappa_0)K(x) + 2M(x) + 2\overline{K(x)} + 2\overline{M(x)}], \quad x \in L \\
&(\sigma_y - i\tau_{xy})_0^+ - (\sigma_y - i\tau_{xy})_0^- = 2\,ihK'(x), \quad x \in L \\
&(u' + iv')_0^+ - (u' + iv')_0^- = \frac{ih}{\mu_0}\,M'(x), \quad x \in L \\
&(u' + iv')_0^+ + (u' + iv')_0^- = \\
&\qquad \frac{1}{\mu_0(1+\varkappa_0)}\,[2\varkappa_0 K(x) + (\varkappa_0 - 1)M(x) - 2\overline{K(x)} - 2\overline{M(x)}], \quad x \in L
\end{aligned} \tag{1.4}$$

where $K(x)$ and $M(x)$ are unknown functions to be determined.

Starting from (1.2), we write the boundary conditions on the edges of the inclusion in the form

$$\begin{aligned}
&(\sigma_y - i\tau_{xy})^+ + (\sigma_y - i\tau_{xy})^- = [\Phi(x) + \Omega(x)]^+ + \\
&\qquad [\Phi(x) + \Omega(x)]^-, \quad x \in L \\
&(\sigma_y - i\tau_{xy})^+ - (\sigma_y - i\tau_{xy})^- = [\Phi(x) - \Omega(x)]^+ - [\Phi(x) - \\
&\qquad \Omega(x)]^- + 2\,K_1(x), \quad x \in L \\
&(u' + iv')^+ + (u' + iv')^- = \frac{1}{2\mu}\,\{\varkappa[\Phi^+(x) + \Phi^-(x)] - [\Omega^+(x) + \Omega^-(x)]\} \\
&x \in L \\
&(u' + iv')^+ - (u' + iv')^- = \\
&\qquad \frac{1}{2\mu}\,\{\varkappa[\Phi^+(x) - \Phi^-(x)] + [\Omega^+(x) - \Omega^-(x)] + 2M_1(x)\}, \quad x \in L
\end{aligned} \tag{1.5}$$

$$K_1(x) = ih\,[\Phi_0'(x) - \Omega_0'(x) + 2\overline{\Phi_0'(x)}]\,\frac{\min(\mu_0, \mu)}{\mu} \tag{1.6}$$

$$M_1(x) = ih\,[\varkappa\Phi_0'(x) + \Omega_0'(x) - 2\overline{\Phi_0'(x)}]\,\frac{\min(\mu_0, \mu)}{\mu_0}$$

Using the dependence (1.1), we obtain the following boundary value problem to determine the piecewise-holomorphic functions $\Phi(z)$ and $\Omega(z)$ with the line of jumps L from (1.4) and (1.5):

$$[\Phi(x)-\Omega(x)]^+ - [\Phi(x)-\Omega(x)]^- = 2\,ihK'(x) - 2\,K_1(x),\quad x\in L \tag{1.7}$$

$$[\varkappa\Phi(x)+\Omega(x)]^+ - [\varkappa\Phi(x)+\Omega(x)]^- = \frac{2\mu}{\mu_0} ihM'(x) - 2M_1(x),\quad x\in L$$

$$[\Phi(x)+\Omega(x)]^+ + [\Phi(x)+\Omega(x)]^- = \frac{2}{(1+\varkappa_0)}[(1-\varkappa_0)K(x)+2M(x)+2\overline{K(x)}+2\overline{M(x)}],\quad x\in L \tag{1.8}$$

$$\varkappa[\Phi^+(x)+\Phi^-(x)] - [\Omega^+(x)+\Omega^-(x)] = \frac{2\mu}{\mu_0(1+\varkappa_0)}[2\varkappa_0K(x)+(\varkappa_0-1)M(x)-2\overline{K(x)}-2\overline{M(x)}],\quad x\in L$$

Solving the linear conjugate problem (1.7), we find

$$\Phi(z) = \frac{h}{\pi(1+\varkappa)}\left[I_k(z)+\frac{\mu}{\mu_0}I_m(z)\right]+\Phi_0(z) \tag{1.9}$$

$$\Omega(z) = \frac{h}{\pi(1+\varkappa)}\left[-\varkappa I_k(z)+\frac{\mu}{\mu_0}I_m(z)\right]+\Omega_0(z)$$

$$I_k(z) = \int_a^b \frac{[K'(t)-K_2(t)]\,dt}{t-z},\quad I_m(z) = \int_a^b \frac{[M'(t)-M_2(t)]\,dt}{t-z}$$

$$K_2(x) = \frac{1}{ih}K_1(x),\quad M_2(x) = \frac{\mu_0}{ih\mu}M_1(x)$$

Using (1.9) and (1.8) we obtain the following system of singular integro-differential, Prandtl-type equations to determine the unknown functions $K(x)$ and $M(x)$:

$$\frac{1}{(1+\varkappa_0)}[(1-\varkappa_0)K(x)+2M(x)+2\overline{K(x)}+2\overline{M(x)}] - \frac{h(1-\varkappa)}{\pi(1+\varkappa)}I_k(x) - \frac{2h\mu}{\pi\mu_0(1+\varkappa)}I_m(x) = \Phi_0(x)+\Omega_0(x),\quad x\in L \tag{1.10}$$

$$\frac{\mu}{\mu_0(1+\varkappa_0)}[2\varkappa_0K(x)+(\varkappa_0-1)M(x)-2\overline{K(x)}-2\overline{M(x)}] - \frac{2h\varkappa}{\pi(1+\varkappa)}I_k(x) - \frac{h\mu(\varkappa-1)}{\pi\mu_0(1+\varkappa)}I_m(x) = \varkappa\Phi_0(x)-\Omega_0(x),\quad x\in L$$

2. Following [2], we seek the solution of the system (1.10) in the form

$$K(x_1) = K_0+K_3(x_1)-\sqrt{1-x^2}\sum_{m=1}^{\infty}\frac{1}{m}X_mU_{m-1}(x) \tag{2.1}$$

$$M(x_1) = M_0+M_3(x_1)-\sqrt{1-x^2}\sum_{m=1}^{\infty}\frac{1}{m}Y_mU_{m-1}(x),\quad |x|\leqslant 1$$

$$x_1 = \frac{b-a}{2}x+\frac{a+b}{2} \tag{2.2}$$

$$K_3(x) = [\Phi_0(x) - \Omega_0(x) + 2\overline{\Phi_0(x)} - A_n + B_n - 2\overline{A_n}]\frac{\min(\mu_0, \mu)}{\mu}$$
$$M_3(x) = [\varkappa\Phi_0(x) + \Omega_0(x) - 2\overline{\Phi_0(x)} - \varkappa A_n - B_n + 2\overline{A_n}]\frac{\min(\mu_0, \mu)}{\mu}$$

Here K_0, M_0, X_m, Y_m are unknown coefficients, $T_m(x)$ and $U_m(x)$ are Chebyshev polynomials of the first and second kinds, respectively.

Substituting (2. 1) into (1. 9), we find expressions for the functions $\Phi(z)$ and $\Omega(z)$

$$\Phi(z) = -\frac{2h}{(b-a)(1+\varkappa)}\sum_{m=1}^{\infty}\left(X_m + \frac{\mu}{\mu_0}Y_m\right)\left[\left(\frac{b-a}{2}\right)\frac{T_m(z_1)}{\sqrt{(z-a)(z-b)}} - U_{m-1}(z_1)\right] + \Phi_0(z) \qquad (2.3)$$

$$\Omega(z) = \frac{2h}{(b-a)(1+\varkappa)}\sum_{m=1}^{\infty}\left(\varkappa X_m - \frac{\mu}{\mu_0}Y_m\right)\left[\left(\frac{b-a}{2}\right)\frac{T_m(z_1)}{\sqrt{(z-a)(z-b)}} - U_{m-1}(z_1)\right] + \Omega_0(z)$$

$$z = \frac{b-a}{2}z_1 + \frac{a+b}{2}$$

Starting from (1. 10) and (2. 1), and following [2], we arrive at an infinite system of linear algebraic equations to determine the expansion coefficients X_m and Y_m

$$\frac{1}{(1+\varkappa_0)}\sum_{m=1}^{\infty}\frac{1}{m}H(m, n)[(1-\varkappa_0)X_m + 2Y_m + 2\overline{X}_m + 2\overline{Y}_m] + C_1X_n + C_2Y_n = D_n \qquad (2.4)$$

$$\frac{\mu}{\mu_0(1+\varkappa_0)}\sum_{m=1}^{\infty}\frac{1}{m}H(m, n)[2\varkappa_0X_m + (\varkappa_0 - 1)Y_m - 2\overline{X}_m - 2\overline{Y}_m] + C_3X_n + C_4Y_n = P_n$$

$$H(m, n) = \begin{cases} 0, & \text{if } m+n \text{ are odd} . \\ \dfrac{1}{(m+n+1)(m+n-1)} - \dfrac{1}{(m-n-1)(m-n+1)} & \text{if } m+n \text{ are even} \end{cases} \qquad (2.5)$$

$$C_1 = \frac{\pi h(1-\varkappa)}{(b-a)(1+\varkappa)}, \quad C_2 = \frac{2\pi h\varkappa\mu}{(b-a)(1+\varkappa)\mu_0}$$

$$C_3 = \frac{2\pi h\varkappa}{(b-a)(1+\varkappa)}, \quad C_4 = \frac{\pi h\mu(\varkappa-1)}{(b-a)\mu_0(1+\varkappa)}$$

$$D_0 = \frac{1}{(1+\varkappa_0)}[(1-\varkappa_0)K_0 + 2M_0 + 2\overline{K}_0 + 2\overline{M}_0]$$

$$P_0 = \frac{\mu}{\mu_0(1+\varkappa_0)}[2\varkappa_0K_0 + (\varkappa_0-1)M_0 - 2\overline{K}_0 - 2\overline{M}_0]$$

$$D_n = \int_{-1}^{1}\Big\{-\Phi_0(x_1) - \Omega_0(x_1) + \frac{1}{(1+\varkappa_0)}[(1-\varkappa_0)K_3(x_1) + 2M_3(x_1) +$$

$$2\overline{K_3(x_1)} + 2\overline{M_3(x_1)}] + D_0\Big\}\sqrt{1-x^2}\,U_{n-1}(x)\,dx$$

$$P_n = \int_{-1}^{1}\Big\{\Omega_0(x_1) - \varkappa\Phi_0(x_1) + \frac{\mu}{\mu_0(1+\varkappa_0)}[2\varkappa_0 K_3(x_1) + (\varkappa_0 - 1)M_3(x_1) -$$

$$2\overline{K_3(x_1)} - 2\overline{M_3(x_1)}] + P_0\Big\}\sqrt{1-x^2}\,U_{n-1}(x)\,dx$$

We assume the following values for D_0 and $\operatorname{Re} P_0$

$$D_0 = (A_n + B_n)\frac{\min(\mu_0, \mu)}{\mu}, \quad \operatorname{Re} P_0 = \operatorname{Re}(\varkappa A_n - B_n)\frac{\min(\mu_0, \mu)}{\mu_0} \tag{2.6}$$

and we find the constant $\operatorname{Im} P_0$ from the condition (Λ is the domain of the inclusion) [3]

$$\operatorname{Re}\int_{\Lambda} z\,[\overline{\Omega}(z) + \Phi(z)]\,dz = 0 \tag{2.7}$$

Taking account of (2.3), after manipulation we obtain from (2.7)

$$\operatorname{Im} X_1 = 0 \tag{2.8}$$

By using the results of [2, 4], it can be shown that the system of linear algebraic equations (2.4) will be quasiregular.

Proceeding in the same manner as was done in [5], the state of stress in the neighborhood of the end of the inclusion can be represented in the polar r, θ coordinate system (Fig. 1) in the form

$$\left\|\begin{matrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{matrix}\right\| = \frac{K_1}{4\sqrt{2r}}\left\|\begin{matrix}5\cos{}^1/_2\theta - \cos{}^3/_2\theta\\ 3\cos{}^1/_2\theta + \cos{}^3/_2\theta\\ \sin{}^1/_2\theta + \sin{}^3/_2\theta\end{matrix}\right\| + \tag{2.9}$$

$$\frac{K_2}{4\sqrt{2r}}\left\|\begin{matrix}-5\sin{}^1/_2\theta + 3\sin{}^3/_2\theta\\ -3\sin{}^1/_2\theta - 3\sin{}^3/_2\theta\\ \cos{}^1/_2\theta + 3\cos{}^3/_2\theta\end{matrix}\right\| +$$

$$\frac{K_3}{4\sqrt{2r}}\left\|\begin{matrix}5\cos{}^1/_2\theta + (1+2\varkappa)\cos{}^3/_2\theta\\ 3\cos{}^1/_2\theta - (1+2\varkappa)\cos{}^3/_2\theta\\ \sin{}^1/_2\theta - (1+2\varkappa)\sin{}^3/_2\theta\end{matrix}\right\| +$$

$$\frac{K_4}{4\sqrt{2r}}\left\|\begin{matrix}-5\sin{}^1/_2\theta + (1-2\varkappa)\sin{}^3/_2\theta\\ -3\sin{}^1/_2\theta - (1-2\varkappa)\sin{}^3/_2\theta\\ \cos{}^1/_2\theta + (1-2\varkappa)\cos{}^3/_2\theta\end{matrix}\right\| + O(r^\circ)$$

Here K_i $(i = 1, 2, 3, 4)$ are stress intensity coefficients which are determined by the formulas

$$K_1^j - iK_2^j = -\frac{2h\mu}{\mu_0(1+\varkappa)}\left(\frac{b-a}{2}\right)^{-1/2}\sum_{m=1}^{\infty}(-1)^{(m+1)(2-j)}Y_m \tag{2.10}$$

$$K_3^j - iK_4^j = -\frac{2h}{(1+\varkappa)}\left(\frac{b-a}{2}\right)^{-1/2}\sum_{m=1}^{\infty}(-1)^{(m+1)(2-j)}X_m$$

($j = 1$ for the end a and $j = 2$ for the end b).

Passing to the limit in (1.10) or (2.4), respectively, as $\mu_0 \to \infty$, $\mu_0 \to 0$, $\mu_0 \to \mu$ and taking account of (1.9), (2.3), (2.6) and (2.8), we obtain the solution of the following problems: for an absolutely rigid inclusion, a pliable inclusion (slit), and a homogeneous strip (beam).

3. A numerical analysis was performed for the following cases: 1) pure bending of a beam by moments M; 2) Deformation of a beam subjected to a uniformly distributed pressure of intensity q along the length.

According to [1], the coefficients A_i, B_i have the following form

$$A_2 = M / (4I), \quad B_2 = 3M / (4I), \quad A_i = B_i = 0 \quad (i = 0, 1, 3, 4, \ldots, n)$$

in the first case and

$$A_0 = q / (24I), \quad A_2 = q(w^2 - 3H^2/5)/(8I), \quad A_3 = -qH^3/(12I)$$
$$B_0 = 7q / (24I), \quad B_2 = q(3w^2 - 11H^2/5)/(8I), \quad B_3 = qH^3/(12I)$$
$$A_i = B_i = 0 \quad (i = 1, 4, 5, \ldots, n), \ldots,$$

in the second case, where $I = 4\tau H^3 / 3$ and $2w$ is the length of the beam.

The dependence of the stress intensity coefficients $K_i' = \sqrt{2} I K_i / (3Ma^{3/2})$ ($i = 1, 2, 3, 4$) on the relative stiffness of the inclusion and the strip $k = \mu_0 / \mu$ is represented in Figs. 2 and 3. The same dependence but only the quantities $K_i' = IK_i / (\sqrt{2} q a^{7/2})$ are given in Figs. 4 and 5. Curves *1* and *2* characterize the stress intensity coefficients $(-K_1')$ and $(-K_3')$ respectively, at the point a, and curves *3* and *4* the K_1' and K_3' at the point b. Let us note that the curves in Figs. 3 and 5 are a continuation of the corresponding curves in Figs. 2 and 4. For the examples under consideration $K_2' = K_4' = 0$. In the first case the calculations are performed for the following values of the parameters in the problem: $h / a = 0.45$, $b / a = 10$, and in the second for $h / a = 0.2$, $b / a = 5$, $H / a = 10$, and $w / a = 10$. For both cases it is considered that the Poisson's ratios equal $\nu = \nu_0 = 1/3$.

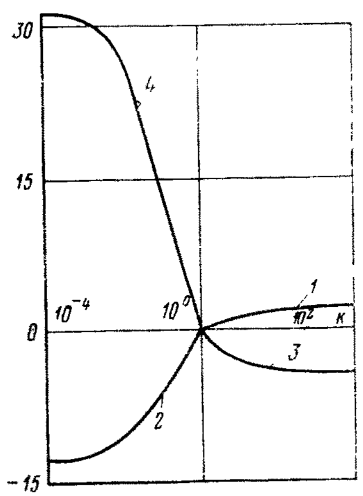

Fig. 2

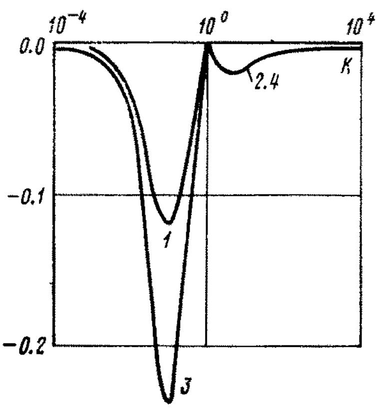

Fig. 3

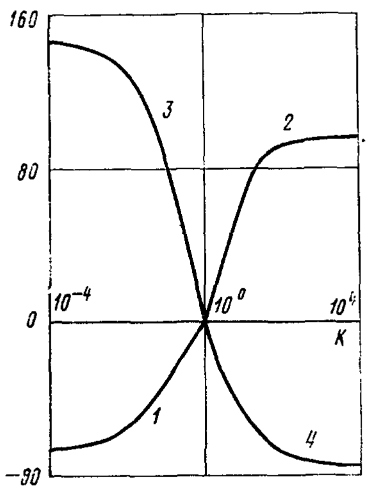

Fig. 4

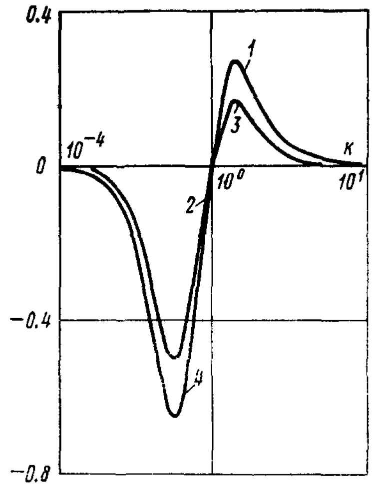

Fig. 5

REFERENCES

1. P a n a s i u k, V. V., Ultimate Equilibrium of Brittle Bodies with Cracks. "Naukova Dumka", Kiev, 1963.

2. M o r a r', G. A. and P o p o v, G. Ia., On the contact problem for a half-plane with finite elastic reinforcement, PMM, Vol. 34, No. 3, 1970.

3. M u s k h e l i s h v i l i, N. I., Some Basic Problems of the Mathematical Theory of Elasticity, (English translation), Groningen, Noordhoff, 1953.

4. S u l i m, G. T., Regularity of some systems of linear algebraic equations. Visn. L'vivs'k. Univ. Ser. Mekhan., Matem., No. 10, 1975.

5. P a n a s i u k, V. V., S a v r u k, M. P., and D a t s i s h i n, A. P., Stress Distribution Around Cracks in Plates and Shells," Naukowa Dumka", Kiev, 1976.

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